On state space representation for linear discrete-time systems in Hilbert spaces

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For a linear continuous-time control system in Hilbert space with state $x(t)$ is associated a discrete-time system where the state variable is $z_k = (x((k+1)h) + x(kh))/2$, with small h. This allows to introduce a discrete derivative $\Delta z_k = (x((k+1)h) - x(kh))/h$. The obtained discrete-time system has structural properties with a similar formulation as continuous system. Stability is equivalent to the fact that the spectrum of the state operator of discrete-time system is in the left half plane, Lyapunov and Riccati equation are similar.
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1 Introduction

We are concerned with systems described by equations

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t), \\ y(t) & = & Cx(t) \end{array} \right. \tag{1}$$

where x, u and y take values in Hilbert spaces X, U and Y respectively. A, B and C are linear operators. B is bounded and A is the infinitesimal generator of a C_0-semi-group of bounded operators $S(t),\ t \geq 0$. The mild solution of the system (1) is given by:

$$x(t) = S(t)x_0 + \int_0^t S(t-\tau)Bu(\tau)\mathrm{d}\tau$$

A direct discretization of the mild solution gives a discrete system with bounded operators

$$F = S(h), \quad G = \int_0^h S(\tau)B\mathrm{d}\tau, \quad H = C$$

respectively for the state, input and output:

$$\left\{ \begin{array}{rcl} x_{k+1} & = & Fx_k + Gu_k, \\ y_k & = & Hx_k, \end{array} \right. \tag{2}$$

where $x_k = x(kh), y_k = y(kh)$ and $u_k = u(kh)$.
Another way to obtain a discrete-time system from (1) is to use the transfer function, say $T(s) = C(sI - A)^{-1}B$. One can introduce (see for example [6, 3] for the case of infinite dimensional systems) a new transfer function $T_d(w)$. Let

$$T_d(w) = T(\frac{w-1}{w+1}). \tag{3}$$

Under the assumption that $T(s)$ is exponentially stable, T_d is holomorphic and stable outside the unit disc. It is well known (see for example [3]) that $T_d(w)$ is the transfer function of a discrete-time system (A_d, B_d, C_d, D_d):

$$T_d(w) = C_d(wI - A_d)^{-1}B_d + D_d,$$

where

$$A_d = (I - A)^{-1}(I + A), \quad B_d = \sqrt{2}(I - A)^{-1}B$$

and

$$C_d = \sqrt{2}C(I - A)^{-1}, \quad D_d = C(I - A)^{-1}B.$$

Some properties of this discrete system are given in [6] (see also [3], pp. 212-213). In particular, the relation of this system and the original one (1) are discussed. In [4, 6] are investigated the problems of realization, exponential and asymptotic stability for the discrete system obtained from $T_d(s)$ which is not rational function. The corresponding state space is obtained by realization techniques. However it is not clear which connection exists between the continuous-time and the discrete-time states.

The relation (3) introduced earlier for the case of finite dimensional systems allows to obtain properties for discrete-time system from continuous-time systems. Since the early fifties, several developments of continuous-time and discrete-time systems were done in parallel ways. This is the consequence of the different formulation of generic control problems and results. However, as both forms of solutions are computable (Lyapunov and Riccati equations, linear quadratic problems, etc.), the difference between formulations does not induce difficulties. The problem induced by these difference appeared in the late 80s up to now, especially through the robust control problems. In fact, the main idea for unification of continuous and discrete time theories used in some specialized problems is the Tustin transform (3).

A recent contribution has been brought by Bergeon [2] in order to extend this idea and to formalize the relation between the continuous and the discrete systems. The author shaw that every problem formulation and every design available in continuous-time domain can be translated, without loss of generality and simplicity, in the discrete-time domain.
Our purpose is to extend this approach to infinite dimensional systems and to study the specificity, if any, of this case. This approach amounts to putting:

$$z_k = \frac{x_{k+1} + x_k}{2} \tag{4}$$

and

$$\Delta z_k = \frac{x_{k+1} - x_k}{h}. \tag{5}$$

where $x_k = x(kh)$ as for the system (2). In this paper the following assumption is made: $h > 0$ is chosen such that the operator $I + S(h) = I + F$ is bounded invertible. The following discrete-time state space system can be associated to (1):

$$\begin{cases} \Delta z_k &= F_d z_k + G_d u_k, \\ y_k &= H_d z_k + E_d u_k, \end{cases} \tag{6}$$

where the operators F_d, G_d, H_d, E_d are defined later (7). In this system, z_k is the "state" and Δz_k is the "derivative". The new term $E_d u_k$ is the consequence of the discretization: strictly proper system become proper. For this system several control problems are discussed. The main results are that this system is a pseudo-continuous version of the continuous-time system (1). It is shown also that several formulations are similar to those of continuous-time system: Lyapunov and Riccati equations, stability and stabilizability conditions.

2 State representation of discrete-time system

In this section, we show how the system (6) is obtained and how this system converge, in some sense, to the system (1).

Theorem 2.1 *Let $x(t)$ be the mild solution of the system (1) and $x_k = x(kh)$ and $u_k = u(kh)$ for $h > 0$ and $k \in \mathbf{N}$ and let z_k and Δz_k be given by (4) and (5). Then z_k is the solution of the equation*

$$\Delta z_k = F_d z_k + G_d u_k$$

and the output y_k is given by

$$y_k = H_d z_k + E_d u_k,$$

where the operators F_d, G_d, H_d and E_d are bounded and expressed by the relations:

$$\begin{array}{ll} F_d = \frac{2}{h}(F - I)(F + I)^{-1}, & G_d = \frac{2}{h}(F + I)^{-1}G, \\ H_d = 2H(F + I)^{-1}, & E_d = -H(F + I)^{-1}G. \end{array} \tag{7}$$

The operator $F + I = S(h) + I$ being bounded invertible by the choice of h.

Proof. Let us choose first h such that $S(h) + I$ is bounded invertible. If $\rho(S(h))$ denote the resolvent set and $\sigma(S(h))$ the spectrum of $S(h)$. Then it is well known [7] that $e^{h\sigma(A)} \subset \sigma(S(h))$, where $\sigma(.)$ is the spectrum of the corresponding operator A. This means that if $\lambda \in \sigma(A)$, and if $h\lambda \neq 2k\pi i$, then $-1 \in \rho(S(h))$ and $S(h) + I$ is bounded invertible. These values of h are said admissible. It is easy to see that h may be chosen arbitrary small. From the definition of z_k and the relation (2) between x_{k+1} and x_k, we get

$$2z_k = Fx_k + Gu_k + x_k = (F+I)x_k + Gu_k,$$

which gives

$$x_k = 2(F+I)^{-1}z_k - (F+I)^{-1}Gu_k. \tag{8}$$

In the same way, we get for Δz_k the relations:

$$\Delta z_k = \frac{1}{h}\left(x_{k+1} - x_k\right) = \frac{1}{h}\left[(F-I)x_k + Gu_k\right].$$

From (8), we obtain

$$\Delta z_k = \frac{2}{h}(F-I)(F+I)^{-1}z_k - \frac{1}{h}(F-I)(F+I)^{-1}Gu_k + \frac{1}{h}Gu_k.$$

As

$$(F-I)(F+I)^{-1}G = (F+I-2I)(F+I)^{-1}G = G - 2(F+I)^{-1}G,$$

this gives

$$\Delta z_k = \frac{2}{h}(F-I)(F+I)^{-1}z_k + \frac{2}{h}(F+I)^{-1}Gu_k.$$

and then

$$\Delta z_k = F_d z_k + G_d u_k$$

with F_d and G_d given by (7).
For the output relation, from (8), we have

$$y_k = Hx_k = 2H(F+I)^{-1}z_k - H(F+I)^{-1}Gu_k,$$

which gives

$$y_k = H_d z_k + E_d u_k,$$

with the needed operators H_d and E_d. ∎

Remark 2.2 *The original continuous-time system is strictly proper:*

$$\lim_{\Re(s)\to\infty} T(s) = 0.$$

The discrete-time system is only proper. If the continuous-time system is proper, i.e. given by

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t), \\ y(t) & = & Cx(t) + Du(t), \end{array}\right. \tag{9}$$

then, in the corresponding discrete-time system, the operator E_d will be given by

$$E_d = D - H(F+I)^{-1}G.$$

This is the case when the output is not modified. One can also consider a new output $\overline{y}_k = Hz_k$, but this means that we observe in fact Hx_{k+1} also and this system is not causal.

The above discrete-time representation may be called pseudo-continuous representation and converge, when $h \to 0$, to the continuous-time system in the sense given by the Theorem 2.3 and related results of the following section.

The above mentioned properties show that this approach is quite different from that of the construction of pseudo-continuous system from discrete system via the Tustin transformation (see [3] for the infinite dimensional case).

Theorem 2.3 *For all $x_0 \in \mathcal{D}(A)$, $x \in X$ and $u \in U$, we have*

$$\lim_{h\to 0} F_d x_0 = Ax_0, \quad \lim_{h\to 0} G_d u = Bu, \quad \lim_{h\to 0} H_d x = Hx, \quad \lim_{h\to 0} E_d u = 0,$$

the value of h being admissible. If the operator A is bounded, then the limits exist in the uniform operator topology.

Proof. Note that

$$F_d = \frac{2}{h}(F-I)(F+I)^{-1} = 2(F+I)^{-1}\frac{F-I}{h}.$$

As A is the infinitesimal generator of the semigroup $S(t)$, we have

$$\lim_{h\to 0} \frac{F-I}{h} x_0 = \lim_{h\to 0} \frac{S(h)-I}{h} x_0 = Ax_0, \tag{10}$$

for all $x_0 \in \mathcal{D}(A)$. On the other hand,

$$\lim_{h\to 0}(F+I)x = \lim_{h\to 0}(S(h)+I)x = 2x \tag{11}$$

for all $x \in X$ because of strong continuity of the semigroup. Then, for sufficiently small h,

$$\|(S(h)+I)x\| \geq \|x\|.$$

This gives

$$\|x\| = \|(S(h)+I)(S(h)+I)^{-1}x\| \geq \|(S(h)+I)^{-1}x\|,$$

which implies $\|(S(h)+I)^{-1}\| \leq 1$. Then

$$\begin{aligned} \|2(S(h)+I)^{-1}x - x\| &\leq \|(S(h)+I)^{-1}\|\|2x-(S(h)+I)x\| \\ &\leq \|2x-(S(h)+I)x\|. \end{aligned}$$

and by (11), this gives

$$\lim_{h\to 0} 2(F+I)^{-1}x = \lim_{h\to 0} 2(S(h)+I)^{-1}x = x,$$

Then, by a simple calculation and using (10), we get, for $x_0 \in \mathcal{D}(A)$,

$$\lim_{h\to 0} F_d x_0 = \lim_{h\to 0} 2(S(h)+I)^{-1}\frac{S(h)-I}{h} x_0 = Ax_0.$$

Consider now the operator $G_d = \frac{2}{h}(F+I)^{-1}G$. For all $u \in U$, we have (see for example [7]):

$$\lim_{h\to 0}\frac{1}{h}G = \lim_{h\to 0}\frac{1}{h}\int_0^h S(\tau)Bu\mathrm{d}\tau = Bu,$$

and then

$$\lim_{h\to 0}\frac{2}{h}(F+I)^{-1}Gu = Bu.$$

The other limits may be calculated in the same way.
If A is bounded, then $S(t) = \mathrm{e}^{At}$ is uniformly continuous, and all the limits are in the uniform operator topology. ∎

This means that the system (F_d, G_d, H_d, E_d) asymptotically closed to the original continuous-time system. This may be also seen by remarking that if $t = kh$, then for an initial condition $x_0 \in \mathcal{D}(A)$ and a control function $u \in C^1$ we have

$$\lim_{h\to 0} z_k = x(t), \quad \lim_{h\to 0}\Delta z_k = \dot{x}(t).$$

If the initial condition is not in $\mathcal{D}(A)$ and $u \in L_p,\ p \geq 1$, the derivative must be understood in the weak sense (see [3, 7]).

3 Stability and stabilizability

3.1 Stability

The first problem under investigation is that of stability. We consider here only exponential stability for continuous-time system and power stability for discrete-time system. Other concepts of stability mat be considered in a similar way.

Definition 3.1 *The system (1) is said exponentially stable if there exists constants $M \geq 1$ and $\alpha > 0$ such that*

$$\|S(t)\| \leq M\mathrm{e}^{-\alpha t}, \quad t \geq 0.$$

The discrete-time system (2) is said power stable if there exists constants $N \geq 1$ and $0 < \gamma < 1$ such that

$$\|F^n\| \leq N\gamma^n, \quad n \in \mathbf{N}.$$

This means that for $u(t) = 0,\ t \geq 0$ (respectively $u_k = 0, k \in \mathbf{N}$) the solution of both systems verify

$$\int_0^\infty \|S(t)x_0\|^2\mathrm{d}t < \infty, \quad \sum_{i=0}^{\infty}\|x_i\|^2 < \infty.$$

As in this case $z_k = \frac{F+I}{2}x_k$, power stability of systems (2) and (6) are equivalent.

Theorem 3.2 *The system (6) is power stable if and only if the spectrum of F_d, noted $\sigma(F_d)$ is in the interior of the left half plane:*

$$\sigma(F_d) \subset \mathbf{C}_{-\beta} = \{s : \Re(s) < -\beta\}.$$

The corresponding Lyapunov equation is

$$F_d^* P + P F_d = -Q,$$

for positive definite linear bounded operator Q.

Proof. It is well known (cf. for example [3, 5, 8]) that the discrete-time system (2) is power stable if and only if $r(F) < 1$, where $r(F)$ is the spectral radius:

$$r(F) = \sup \{|\lambda| : \lambda \in \sigma(F)\}.$$

The transformation

$$F \mapsto F_d = \frac{2}{h}(F - I)(F + I)^{-1}$$

maps in the same way the spectrum of F:

$$\varphi : \lambda \mapsto \mu = \frac{2}{h}(\lambda - 1)(\lambda + 1)^{-1}, \quad \lambda \neq -1.$$

The spectrum of F is in the interior of the unit ball if and only if $\sigma(F_d) \subset \mathbf{C}_{-\beta}$, because the application φ maps the open unit ball into the open left half plane. On the other hand the condition $\sigma(F_d) \subset \mathbf{C}_{-\beta}$ is equivalent to the existence of positive solution of the Lyapunov equation [3]: $F_d^* P + P F_d = -Q$, with self-adjoint, positive definite operator Q. ∎

3.2 Stabilizability

The above result on stability induce similar criterion on stabilizability.

Definition 3.3 *The system (1) is said stabilizable iff there exists a linear bounded operator K such that the semigroup generated by $A + BK$ is exponentially stable. The system (6) is power stable iff there exists a linear bounded operator K_d such that $F_d + G_d K_d$ is power stable.*

There are several conditions of exponential and power stabilizability. All the known conditions may be extended to the pseudo-continuous system (6) provided that $F_d + G_d K_d$ is power stable iff $\sigma(F_d + G_d K_d) \subset \mathbf{C}_{-\beta}$ for some positive β. In particular, the stabilizability condition may be formulated via the solution of a Riccati equation.

Theorem 3.4 *The system (6) is stabilizable if and only if there exist a positive operator P_d such that:*

$$F_d^* P_d + P_d F_d - P_d G_d G_d^* P_d + Q = 0,$$

and the stabilizing feedback is given by $K_d = -G_d^* P_d$. *The relation between the feedback* K_d *and the feedback stabilizing the discrete-time system (2), say* K, *is given by* $K_d = 2K(F+GK+I)^{-1}$, *and* $K = (2I - K_d G)^{-1} K_d (F+I)$, *under the condition that* $2 \in \rho(K_d G)$, *where* $\rho(.)$ *is the resolvent set of the given operator.*

PROOF. The condition of power stability implies the condition of stabilizability using the Riccati equation (cf. [9]).
Suppose that K is the stabilizing feedback for the system (2), then $u_k = Kx_k$ is the stabilizing control and $x_{k+1} = (F+GK)x_k$ is the closed loop state. Then

$$2z_k = (F+GK)x_k + x_k,$$

which gives

$$x_k = 2(F+GK+I)^{-1} z_k,$$

the bounded invertibility of the operator $F+GK+I$ is garanted by the stability condition of $F+GK$. Hence,

$$u_k = 2(F+GK+I)^{-1} z_k,$$

is the stabilizing feedback for the pseudo-continuous system (6). In an analogous way, under the assumption that $2 \in \rho(K_d G)$, one obtains

$$K = (2I - K_d G)^{-1} K_d (F+I),$$

which ends the proof. ∎

Hence, the Riccati equation for the system (6) is of the same form as for the continuous-time system. Note that as A and A^* are not defined on all the space X, the corresponding Lyapunov and Riccati equations are given on $\mathcal{D}(A)$ (see [3, 9]).

4 Further control problems

The approach developed in Section 2 and 3 may be also extended to other control problems: linear quadratic optimal control problem, detectability, asymptotic observers, etc.

For the problem of detectability and asymptotic observers, the results can be obtained from Section 3 by duality. All the calculation are the same.

For the linear quadratic optimal control problem one can follow the example considered in [3]. A continuous-time system (A,B,C,D) is induced from a discrete-time system (A_d,B_d,C_d,D_d) by the relations:

$$\begin{array}{ll} A = (A_d - I)(A_d+I)^{-1}, & B = \sqrt{2}(A_d+I)^{-1}B_d, \\ C = \sqrt{2}C_d(A_d+I)^{-1}, & C = D_d - C_d(A_d+I)^{-1}B_d, \end{array} \tag{12}$$

and the linear quadratic optimal problem is considered for the continuous-time system (A, B, C, D). It is shown that the optimal solution is obtained via a continuous type Riccati equation.

The same calculation may be made for our pseudo-continuous system (6), and as in the finite dimensional case [2], one can obtains similar formulation for the LQ problem.

5 The input-output relation

In this section we show that the transfer function may be calculated in the same way as for the continuous-time system using the state-space expression of the pseudo-continuous system.

Let $\mathcal{L}$ denote the discrete Laplace transform (in fact the so called z-transform). By $\mathcal{L}(x_k)$ we mean the Laplace transform of the sequence $\{x_0, x_1, \dots\}$. Assume that the initial condition $x_0 = 0$. Then we have $\mathcal{L}(x_{k+1}) = \zeta\mathcal{L}(x_k)$, where ζ is the discrete Laplace variable, and

$$\mathcal{L}(z_k) = \frac{\zeta+1}{2}\mathcal{L}(x_k), \quad \mathcal{L}(\Delta z_k) = \frac{\zeta-1}{h}\mathcal{L}(x_k).$$

From both relations, we obtain

$$\mathcal{L}(\Delta z_k) = \frac{2}{h}\frac{\zeta-1}{\zeta+1}\mathcal{L}(z_k).$$

Putting

$$\omega = \frac{2}{h}\frac{\zeta-1}{\zeta+1}, \tag{13}$$

we get

$$\mathcal{L}(\Delta z_k) = \omega\mathcal{L}(z_k),$$

which gives the Laplace transform of the pseudo-derivative. Applying this calculus to the pseudo-continuous system (6), leads to

$$\mathcal{L}(y_k) = \Big[H_d(\omega I - F_d)^{-1}G_d + E_d\Big]\,\mathcal{L}(u_k).$$

This means that the input-output relation is given by the transfer function

$$\Theta(\omega) = H_d(\omega I - F_d)^{-1}G_d + E_d.$$

The relation (13) between the Laplace variables of the discrete-time system (2) and the pseudo-continuous system (6) is also a Tustin transform like the transform (3), but is applied to the exact discrete system (2) in order to obtain a (pseudo) continuous-time system which is also exact, closed to the continuous-time system (1) and with similar properties. Hence, this approach is quite different from the classical one.

6 Conclusion

From a continuous-time system and its direct discretized system was obtained a pseudo-continuous time system with properties similar to continuous system. Several control problems have the same formulation and characterization for the pseudo-continuous and continuous-time systems. This allows to have the same framework for continuous-time and this kind of discrete-time system. Then, results obtained for continuous system may be extended in an analogous way to the pseudo-continuous systems where the state remains discrete.

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